

A FORMULA FOR THE SHAKURA-SUNYAEV TURBULENT VISCOSITY PARAMETER

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ABSTRACT

A formula for the Shakura-Sunyaev α -parameter is proposed in terms of the growth rate of the unstable modes driving turbulence. In the case of convective instability in a differentially rotating disk, α is given in terms of the parameters of the disk, namely T , R , ρ , and Ω .

Subject headings: convection — stars: accretion — turbulence

I. INTRODUCTION

In the absence of a complete theory, turbulence is traditionally simulated in almost all astrophysical studies by the introduction of a turbulent viscosity ν_t

$$\nu_t = \xi v_t l_t, \quad (1)$$

where l_t and v_t are typical turbulent lengths and turbulent velocities and ξ an unknown parameter. Since most of the energy is contained in the largest eddies, equation (1) is further written as

$$\nu_t = \alpha c_s H, \quad (2)$$

where c_s is the speed of sound, H some characteristic scale height of the problem, and α an unknown parameter. Equation (2) has been widely used in the study of accretion disks (Shakura and Sunyaev 1973; Shakura, Sunyaev, and Zilitinkevich 1978; Pringle 1981), and the parameter α has come to be known as the Shakura-Sunyaev α -parameter.

It is the aim of this *Letter* to propose a formula for α in terms of the growth rate $n(k)$ of the unstable modes of the physical mechanism that generates turbulence. As a particular example, we discuss turbulent convection, whose relevance to the primitive solar nebula was first discussed by Lin and Papaloizou (1980) and further worked out by Lin (1981), Lin and Bodenheimer (1982), Cassen and Summers (1983), and Cameron (1983). The role of convection in accretion disk models has been discussed by Shakura, Sunyaev, and Zilitinkevich (1978), Livio and Shaviv (1977), Liang (1977), Vila (1978), Tayler (1980), and Smak (1982). Our main result, equations (9) and (12), is that α is a well-defined function of the parameters of the problem, i.e., T , R , ρ , and Ω .

II. TURBULENCE MODEL

A major difficulty in constructing an analytical model for fully developed turbulence lies in the well-known closure problem whereby the equation for $\langle v^n \rangle$ (v is the fluctuating or turbulent velocity and $\langle \rangle$ denotes ensemble average) depends on terms of the form $\langle v^{n+1} \rangle$ which in turn satisfy an equation involving $\langle v^{n+2} \rangle$, thus giving rise to an infinite chain of connected equations. For $\langle v^2 \rangle$, one has the well-known en-

ergy equation (Batchelor 1970)

$$\varepsilon(k) = \left[\nu + \int_k^\infty \Psi(k) dk \right] \int_0^k 2k^2 F(k) dk, \quad (3)$$

$$v^2(k) \equiv 2 \int_k^\infty F(k) dk.$$

The energy (per unit mass and time) $\varepsilon(k)$ fed into the system at the scale k is partly dissipated by viscous forces, $\nu(\nabla v)^2 \sim \nu k^2 v^2$, and partly transferred by the nonlinear terms $\langle uvv \rangle$ to wavenumbers higher than k . Following the original suggestion by Heisenberg, the nonlinear term is written in equation (3) as the product of two factors. The first represents the loss of energy by the eddies in the interval $0-k$, and the second represents the redeposition of the same energy into the remaining interval $k-\infty$. Since Heisenberg's suggestion amounts to a renormalization of the molecular viscosity ν , it has become customary to introduce the notion of turbulent viscosity $\nu_t(k)$,

$$\nu_t(k) = \int_k^\infty \Psi(k) dk. \quad (4)$$

It is important however to stress that while ν is constant, $\nu_t(k)$ is not. Moreover, ordinary viscosity dissipates energy into heat, whereas the nonlinear term, or equivalently $\nu_t(k)$, does not: it merely transfers energy from the large eddies to the smaller ones. (The fact that energy as a whole is conserved by the nonlinear terms can be seen from the fact that the integral over all k values of the nonlinear terms is zero.)

The goal of any theory of turbulence is that of providing a "closure," i.e., a relation between $\Psi(k)$ and $F(k)$, so that equation (3) can be solved for the spectral function $F(k)$. The only existing successful model of turbulence is that of Heisenberg and Kolmogorov (hereafter HK, Batchelor 1970). We briefly review this model in order to point out why it cannot be used to estimate α in equation (2). To solve equation (3), one needs two ingredients, $\varepsilon(k)$ and $\Psi(k)$. Broadly speaking, a turbulent medium can be thought of as composed of (a) large eddies and (b) medium to small eddies. The latter originate from the breakup of the former, and being a secondary product, these eddies can no longer be expected to

carry the imprint of the detailed nature of the stirring mechanism. Different stirring mechanisms yield different forms of $\varepsilon(k)$, i.e., they pump energy to a given scale at different rates. By construction, the HK model is a model for those relatively small eddies sufficiently removed in k space from the energy source to feel all the same energy input, $\varepsilon(k) = \varepsilon = \text{constant}$. This choice also has implications on the form of $\Psi(k)$. In fact, the restriction to eddies far removed from the energy source implies that the dimensionful quantities that make up $\varepsilon(k)$ (rotation, magnetic fields, convection, etc.) are all lumped into a single structureless constant ε . The only two remaining variables are $F(k)$ and k , and so $\Psi(k) = \gamma F^{1/2}(k) k^{-3/2}$. With this, equation (3) can be solved (Batchelor 1970), and the results are ($\nu = 0$)

$$F(k) = (8\varepsilon/9\gamma)^{2/3} k^{-5/3}, \quad \nu_i(k) = \xi \nu_i l_i, \quad (5)$$

with $\xi = \gamma\sqrt{3}/4\pi$. Since the assumptions underlying the HK model restrict its application to the small eddies, equation (5) cannot be used to estimate α in equation (2) which refers to the largest eddies.

We must therefore abandon the HK model and derive an expression for $\varepsilon(k)$ that explicitly takes into account the nature of different feeding mechanisms. Using the work of Ledoux, Schwarzschild, and Spiegel (1961), we obtain

$$\varepsilon(k) = 2 \int_{k_0}^k [n(k) + \nu k^2] F(k) dk, \quad (6)$$

where $n(k)$ is the growth rate of the unstable modes that feed energy into the system, eventually generating turbulence (k_0 is the wavenumber of the largest eddy which in eq. [3] was taken to be zero). Equation (6) is clearly very general since all the features of the specific instability at work in a given problem are included in $n(k)$. Equation (3) then becomes

$$\int_{k_0}^k F(k) n(k) dk = \nu_i(k) \int_{k_0}^k F(k) k^2 dk. \quad (7)$$

To solve equation (7) for $F(k)$, one needs a new closure since the HK closure is no longer applicable to this region. Two of us (Canuto and Goldman 1984) have recently proposed a closure formula that allows an analytic solution of equation (7) in terms of $n(k)$. For the case of turbulent convection, the results compared very favorably with laboratory data. However, for the limited purpose of calculating α , one does not need a full theory of turbulence, i.e., the form of $F(k)$. In fact, since we are interested only in ν_i at $k = k_0$, the explicit form of $F(k)$ is irrelevant since both sides of equation (7) are linear in $F(k)$ and at $k = k_0$, $F(k_0)$ cancels out of the equation. We therefore suggest that, for this specific problem, we can regard equation (7) not as an equation for $F(k)$ but as an algebraic relation giving $\nu_i(k)$ at, and only at, $k = k_0$. Therefore, taking the limit for $k \rightarrow k_0$, equation (7) yields

$$\nu_i(k_0) = \frac{n(k_0)}{k_0^2}, \quad (8)$$

which is our main result.

For simplicity, let us approximate the actual geometry by a plane-parallel model with the z -component of k , k_z , satisfying the boundary conditions $k_z L = n\pi$ ($n = 1, 2, \dots$), where $0 \leq z \leq L$. We have $k_0 L = \pi\sqrt{1+x}$, $x \equiv (k_x^2 + k_y^2)/k_z^2$. Using equations (8) and (2), we derive

$$\alpha = \frac{1}{\pi^2(1+x)} \left(\frac{H}{c_s} \right) n(k_0) \left(\frac{L}{H} \right)^2. \quad (9)$$

III. CONVECTIVE TURBULENCE

A general treatment of the stability conditions of a differentially rotating flow has been given by Goldreich and Schubert (1967, hereafter GS). The growth rate $n(k)$ satisfies in general a fifth-order polynomial given by equations (17)–(22) of GS. Depending on the specific problem at hand, simplifications are possible. For example, in the cases where $\Delta\rho/\rho \ll 1$ and $\partial\rho/\partial z \ll k_z\rho$, the first three terms of the continuity equation (eq. [20] of GS) can be neglected, with the result that the growth rate satisfies a third-order equation. Assuming further that $\Delta\chi/\chi \ll 1$, where χ is defined below, the final result can be written as (for a Keplerian disk)

$$\frac{L^4}{\nu\chi} (n + \chi k^2)(n + \nu k^2) = \frac{\mathcal{R}\chi}{1+x} \left(1 - \frac{\sigma T^*}{\mathcal{R}\chi} \frac{n + \chi k^2}{n + \nu k^2} \right). \quad (10)$$

Here, χ is the thermometric conductivity (related to the thermal conductivity K by $c_p\rho\chi = K$), ν is the kinematic viscosity, \mathcal{R} is the Rayleigh number $= g_z\bar{\alpha}BL^4/\chi\nu$, $\bar{\alpha}$ is the coefficient of thermal expansion, β is the temperature gradient excess over the adiabatic gradient, g_z is the z component of gravity ($g_x = g_y = 0$), $\Omega = (0, 0, \Omega)$ is the rotation vector, $T^* = 4L^4\Omega_*^2/\nu^2$, where $\Omega_*^2 = \Omega^2(1 + R\Omega'/2\Omega)$ with $\Omega' = d\Omega/dR$, is the effective Taylor number, and $\sigma = \nu/\chi$ is the Prandtl number. Introducing the dimensionless variable

$$N \equiv \frac{n(k_0)}{(g_z\bar{\alpha}\beta)^{1/2}}$$

to be taken as the real, positive part of the solutions of equation (10) for $k = k_0$,

$$(N + N_0)(N + \sigma N_0) = \frac{x}{1+x} \left(1 - \frac{\sigma T^*}{\mathcal{R}\chi} \frac{N + N_0}{N + \sigma N_0} \right),$$

$$N_0 \equiv \frac{(1+x)\pi^2}{(\mathcal{R}\sigma)^{1/2}}, \quad (11)$$

equation (9) can be rewritten as ($g_z = gz/R$)

$$\alpha = \frac{1}{\pi^2} (z\bar{\alpha}\beta)^{1/2} \left(\frac{L}{H} \right)^2 \frac{N}{(1+x)} \frac{H}{c_s} \left(\frac{g}{R} \right)^{1/2}. \quad (12)$$

To evaluate α , it is useful to express $\mathcal{R}\sigma$ in terms of the physical variables of the problem. Since in the case of radia-

tive conduction, $\chi = 4acT^3(3k_{\text{op}}c_p\rho^2)^{-1}$, where k_{op} is the opacity, we derive

$$\mathcal{R}\sigma = 1.45 \times 10^{33} \left(\frac{M}{M_{\odot}} \right) (z\bar{\alpha}\beta) \frac{(\rho L)^4}{(RT^2)^3} c_p^2 k_{\text{op}}^2. \quad (13)$$

Equations (11)–(13) provide the determination of α .

IV. EFFECT OF ROTATION ON ν_i

While the effect of rotation on ν_i can only be fully quantified once equation (11) is solved for different values of the parameters σ , $\sigma T^*/\mathcal{R}$, and $\mathcal{R}\sigma$, we have chosen $\sigma = 0$ and $\sigma T^*/\mathcal{R} = (z\bar{\alpha}\beta)^{-1} = 2$ to illustrate the reduction in the value of N due to rotation. The results are shown in Figure 1, where we plot $N/(1+x)$ versus x for $\mathcal{R}\sigma = 10^4, 10^{10}$. For the first value of $\mathcal{R}\sigma$, convection sets in for any x , although the maximum value is reduced by a factor of 5 with respect to the $\Omega = 0$ case. For $\mathcal{R}\sigma = 10^{10}$, a new feature appears, namely equation (11) has no solutions (no convection) until $x > \sigma T^*/\mathcal{R}$. This feature can also be seen analytically by taking $\mathcal{R}\sigma \rightarrow \infty$ in equation (11). This gives

$$N = \left[\frac{x}{1+x} \left(1 - \frac{\sigma T^*}{x\mathcal{R}} \right) \right]^{1/2} \quad (14)$$

which clearly shows that x must be greater than $\sigma T^*/\mathcal{R}$ for N to be real.

For the two particular cases shown in Figure 1, we can estimate the value of α using equation (12) and $dp/dz = -g_z\rho$, $p \sim \rho^{\gamma}$, i.e., $(H/c_s)(g/R)^{1/2} = (2/\Gamma)^{1/2}$. For the maximum value of $N/(1+x)$ and for $\Gamma = \frac{5}{3}$, we obtain

$$\begin{aligned} \alpha &= 0.016(L/H)^2 & (\mathcal{R}\sigma = 10^{10}), \\ \alpha &= 0.009(L/H)^2 & (\mathcal{R}\sigma = 10^4). \end{aligned} \quad (15)$$

Since different values of $\sigma T^*/\mathcal{R}$ yield different values of α , one cannot give a unique value for α , but the latter must be determined consistently with the other equations for the disk.

V. CONVECTIVE FLUXES

For a complete description of the disk, one needs one more quantity, the convective flux Φ as a function of Ω and $\mathcal{R}\sigma$. A full discussion of this topic is outside the scope of this Letter, but the following remarks may be of interest.

Lacking a complete theory, the most general form presently available is the one derived with the (one-mode) mixing-length theory (MLT; Spiegel 1963; Gough 1976, 1978), which gives

$$\Phi = c_p \rho \beta \chi_i, \quad \chi_i/\chi = \frac{A}{1+x} N^3 (\mathcal{R}\sigma)^{1/2}, \quad (16)$$

with the coefficient $A = 0.65$. This equation has been used in the study of the primitive solar nebula assuming constant N and x (Lin and Papaloizou 1980).

The present analysis and Figure 1 show that $N/(1+x)$ is a nonmonotonic function of x and $\mathcal{R}\sigma$, indicating that the effect

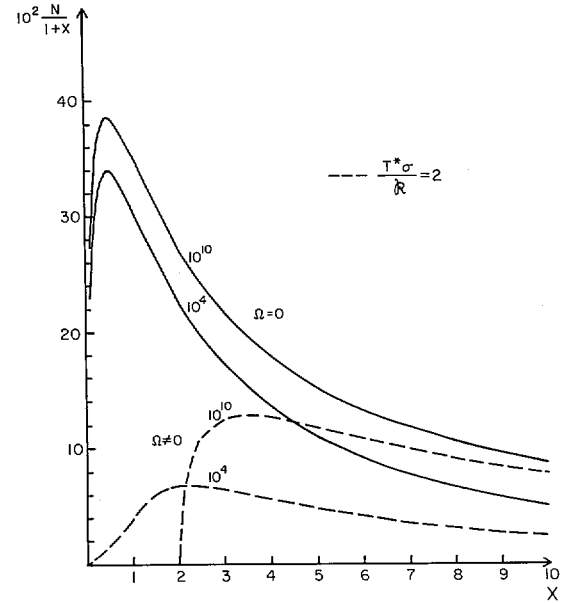


FIG. 1.—Solutions of equation (11) for $\nu = 0$, with and without rotation. The numbers on each curve are the values of $\mathcal{R}\sigma$. The quantity $\sigma T^*/\mathcal{R} = (z\bar{\alpha}\beta)^{-1}$ has been taken as equal to 2.

of rotation on N is a complex one that can hardly be accounted for by changing one of the scale parameters of the problem. For this reason, the results based on this approximation (Lin 1981; Lin and Papaloizou 1980) must be taken with caution.

A formalism more general than the one-mode MLT approximation and one which is valid for arbitrary values of Ω and $\mathcal{R}\sigma$ has recently been developed by Canuto and Goldman (1984).

VI. STABILITY OF THE DISK

Many papers have discussed the stability of a stationary accretion disk against viscosity and temperature perturbations (see Pringle 1981, and references therein). It was, however, noted by Pringle that because of the arbitrariness in defining ν_i , it is not clear whether the instabilities are real or just follow from an improper viscosity law. Within the present framework, the arbitrariness in the definition of ν_i is no longer present. Once an $n(k)$ is chosen, the disk equations can be solved, the numerical value of the function $\nu_i = \nu_i(T, R, \Omega)$ found, and the conditions for thermal and viscosity stability (eqs. [7.4] and [7.6] of Pringle 1981) checked.

With this procedure, the global stability requirements become a tool to assess the reality of the instability thought to originate turbulence in the disk.

VII. OTHER TYPES OF INSTABILITIES

Since equation (2) is, in the ultimate analysis, a simple parameterization of turbulent viscosity independently of whether it is used in accretion disks, it may be useful to quote forms for $n(k)$ corresponding to instabilities other than convection. Parker-like instabilities (Parker 1966) have recently been generalized by Elmegreen (1982) to include self-gravity. Thermal-convective instabilities have been worked out by

Field (1965), Defouw (1970*a, b*), and Goldsmith (1970), who provide several expressions for $n(k)$. Hide (1955), Skumanich (1955), Spiegel and Unno (1962), and Chandrasekhar (1961) have provided analytic expressions for $n(k)$ corresponding to Rayleigh-Taylor instabilities; and finally Chandrasekhar (1961) has provided graphs and tables for $n(k)$ corresponding to convection in the presence of magnetic fields and rotation and for the case of Helmholtz-Kelvin instabilities.

VIII. CONCLUSIONS

We have derived a general expression for α in terms of the growth rate of the unstable modes driving the turbulence,

equation (9). This general result is then applied to the case of convective turbulence, equation (12).

While it has been customary to treat α as a constant parameter, in recent years several authors (Williams 1980; Frank and King 1981; Smak 1982) have shown that self-consistent convective solutions for disks require different α values for different values of T and R , as our formalism suggests. The present analysis does offer a scheme to compute α and, consequently, the amount of heat generated by turbulent viscosity.

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